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Effective Lagrangian due to Heavy Quarks in Quantum Chromodynamics

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ABSTRACT

We extend our previous analysis of heavy particle effects in low energy light particle sector to quantum chromodynamics. The central technique used is Ward-Takahashi identities due to Becchi-Rouet-Stora transformations. Some discussion of isolation of mass singularities is given. An important result in our approach, as before, is to give precise meaning to the notion of an effective Lagrangian (which is renormalized) with calculable effective coupling constants. The counter terms for the effective local vertices in the Lagrangian are self-generated by the theory, while the effective coupling constants obey a set of Callan-Symanzik-like equations. The present article is self-contained.

I. INTRODUCTION

In several articles recently written by us,[1] a formalism was laid, through which one can systematically and reliably investigate effects of heavy particles on light particle sector in low energy region. This is done via factorized local operators with calculable universal coefficients. A precise definition of an effective Lagrangian is thus given. Although these same results are anticipated in theories without spontaneously broken symmetry, the concrete model studied was quantum electrodynamics (QED) with heavy muons and light electrons and photons.

In view of the heightened promise of quantum chromodynamics (QCD) as a viable theory of strong interaction, it is of some urgency that we should explicitly demonstrate the validity of our technique in this arena. One can then efficiently assess effects of heavy quarks in processes below their production thresholds. This we have succeeded in doing. The formal aspects will be given in the present article; in a sequel, we shall report on a detailed calculation with application to e^+e^- total annihilation cross-section.

In I, it is established that if M is the generic mass of the heavy particles, if all the external momenta of the proper amputated n light particle Green's function

Γ^n are small compared to M, and if the scale μ at which it is subtracted for renormalization is also small ($\mu \ll M$), then we have[2,3]

$$\Gamma^n_{\text{full theory}} = \Gamma^n_{\text{light theory}} + \frac{1}{M^2} \sum_i C_i \, \Gamma^n_{\text{light theory}}(O_i) + O(1/M^4) \tag{I-1}$$

where 'full theory' means that the defining Lagrangian includes both the heavy and the light fields, and 'light theory' has only light components. C_i's have all the large mass dependence in the form of $\ln(M^2)$ and are calculable via a set of Callan-Symanzik-like equations.[4] O_i's are (integrated) local operators whose densities have naive dimensions less than or equal to six.

There are two elements in I on which variation and/or improvement in argument seem desirable in extending our analysis to QCD. It should be helpful to have a procedure which encompasses both gauge invariance and renormalization simultaneously. (In I, these two aspects were implemented somewhat independently.) We would also like to give an explicit discussion of how infrared and collinear singularities are isolated in the context of QCD.

A major difference between QED and QCD is their distinct gauge transformation properties. For the Abelian gauge theory, the Ward-Takahashi (W-T) identities are linear and simple. This allowed us in I to impose gauge invariance with ease. In fact, O_i's are all manifestly gauge invariant in QED.

When it comes to QCD, the non-Abelian nature of the gauge transformations leads to W-T identities which are essentially non-linear. This makes an attack on the present problem via the Zimmermann's[5] analysis as in I impractical. As it turns out, because we are interested only in $1/M^2$ effects, it is possible to linearize the W-T identities in loop expansion. In point of fact, the W-T identities together with power counting are sufficient to establish Eq. (1) in QCD. Needless to say, in hind-sight, one can do likewise for QED.

Another technical point which needs some modification in QCD is the renormalization procedure. In I, the fermions are assumed to be massive. There is no collinear singularity and we conveniently choose to renormalize the operators at zero external momenta, which is the most natural point in the context of Zimmermann's analysis. This we cannot do in QCD. We must renormalize at some Eucledean point μ, which will inevitably induce operators of dimension four together with those of dimension six. How these lower dimension operators are renormalized will have to be considered.

The relevant observation to make in order to by-pass this apparent complication is the following: it should be noted that the right hand side of Eq. (1) is merely an algebraic rearrangement of a renormalized series, i.e. $\Gamma^n_{\text{full theory}}$. In other words, we are adding and subtracting the same quantities order by order in $\Gamma^n_{\text{full theory}}$. This is how the effective local operators O_i are induced and the coefficients C_i extracted. Thus, we may choose as we please how to render O_i finite, irrespective of how Γ^n are renormalized. The normalization conditions on $\Gamma^n_{\text{full theory}}$ and $\Gamma^n_{\text{light theory}}$ are all we need to define the renormalization procedure.

The plan of this paper is as follows: In the next section, we shall briefly review the Becchi-Rouet-Stora (BRS) transformation.[6] The aim is to establish notations and to derive the W-T identities.

In Section III, we shall assume the validity of certain power counting results, which can be straightforwardly inferred from our argument in I. Then, we shall derive Eq. (1) inductively by loop expansion. We shall see that the local operators are precisely the set dictated by W-T identities. Some remarks on infrared and collinear singularity factorization will be made here.

In Section IV, we shall write down a set of renormalization group equations for C_i's. Only massless light quarks will be considered.

A brief conclusion will be given in Section V.

II. WARD-TAKAHASHI IDENTITIES

In this section we shall develop notations for QCD and discuss BRS transformations, which will give us a set of local W-T identities.

The hermitian generators for the fundamental representation in SU(N), to which the quark fields $\psi_\alpha (\alpha = 1, N)$ belong, are $\lambda_a/2$ $(a = 1, N^2-1)$. They satisfy the commutation relation

$$[\lambda_a/2, \lambda_b/2] = 2i\, f_{abc} \frac{\lambda_c}{2} \qquad \text{(II-1)}$$

where f_{abc} are the totally antisymmetric real structure constants. We introduce the antihermitian matrices

$$\tau_a = \lambda_a/2i, \qquad \text{(II-2)}$$

to form the matrix fields

$$\begin{aligned} \hat{A}_\mu &= \hat{A}^a_\mu \tau_a, \\ \hat{F}_{\mu\nu} &= \hat{F}^a_{\mu\nu}\tau_a, \\ \hat{c} &= \hat{c}_a \tau_a, \quad \hat{\bar{c}} = \hat{\bar{c}}_a \tau_a \end{aligned} \qquad \text{(II-3)}$$

where $\hat{c}_a$ and $\hat{\bar{c}}_a$ are, respectively, the ghost and the antighost fields. The covariant derivative is

$$\hat{D}_\mu = \partial_\mu + g_o \hat{A}_\mu \qquad \text{(II-4)}$$

The QCD Lagrangian in linear gauges is

$$\begin{aligned} \mathcal{L}_{eff} = {} & \frac{1}{2}\,\mathrm{Tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}) + \hat{\bar{\psi}}\,(i\not{D} - m_o)\hat{\psi} \\ & + \frac{1}{\alpha_o}\,\mathrm{Tr}(\partial_\mu \hat{A}^\mu)^2 + 2\,\mathrm{Tr}(\hat{\bar{c}}\,\partial_\mu\,[\hat{D}^\mu, \hat{c}]) \end{aligned} \qquad \text{(II-5)}$$

where the trace is taken over the internal symmetry. We put carets to indicate operator fields. We need not assign an index for flavors at this stage. This Lagrangian is invariant under the BRS transformations[6]

$$\begin{aligned}
&\delta_\lambda \hat{A}_\mu = [\hat{D}_\mu, \hat{c}]\delta\lambda \ , \\
&\delta_\lambda \hat{\psi} = g_o \ \hat{c} \ \hat{\psi} \ \delta\lambda, \ \delta_\lambda \hat{\bar{\psi}} = g_o \ \hat{\bar{\psi}} \ \hat{c} \ \delta\lambda, \\
&\delta_\lambda \hat{c} = \frac{1}{2} g_o \ [\hat{c}, \hat{c}] \ \delta\lambda, \\
&\delta_\lambda \hat{\bar{c}} = \frac{1}{\alpha_o} \partial_\mu \hat{A}^\mu \ \delta\lambda
\end{aligned} \tag{II-6}$$

where $\delta\lambda$ is an anticommuting global gauge function which carries a ghost number -1.

It proves convenient to introduce sources into the theory to facilitate construction of Green's functions and discussion of gauge properties. Thus, we write[7]

$$\mathcal{L} = \mathcal{L}_{eff} + \mathcal{L}_{s.} + \mathcal{L}_{c.s.} \tag{II-7}$$

where

$$\begin{aligned}
\mathcal{L}_{s.} = \ & J^a_\mu \hat{A}^\mu_a + \bar{\eta}_\alpha \ \hat{\psi}_\alpha + \hat{\bar{\psi}}_\alpha \ \eta_\alpha \\
& + \bar{\xi}_\alpha \ \hat{c}_a + \hat{\bar{c}}_a \ \xi_a
\end{aligned} \tag{II-8}$$

contains sources for the primary fields, and

$$\begin{aligned}
\mathcal{L}_{c.s.} = \ & - K^a_\mu \ [\hat{D}^\mu, \ \hat{c}]_a + L_a \frac{1}{2} g_o \ [\hat{c}, \hat{c}]_a \\
& + \bar{m} \ g_o \ \hat{c} \ \hat{\psi} + g_o \ \hat{\bar{\psi}} \ \hat{c} \ m
\end{aligned} \tag{II-9}$$

has sources which induce composite operators appearing in BRS transformations of Eq. (II-6).

The generating functional is defined as

$$Z = \int d\hat{A}\, d\hat{c}\, d\hat{\bar{c}}\, d\hat{\psi}\, d\hat{\bar{\psi}}\, \exp\,(i/d^4x\, \mathcal{L}) \tag{II-10}$$

while the connected generating functional is

$$W = -i\, \ell n\, Z \tag{II-11}$$

After identifying the classical fields (without carets) as

$$A_\mu^a = \frac{\delta W}{\delta J_a^\mu},$$

$$\psi_\alpha = \frac{\delta W}{\delta \bar{\eta}_\alpha}, \quad \bar{\psi}_\alpha = -\frac{\delta W}{\delta \eta_\alpha},$$

$$c_a = \frac{\delta W}{\delta \bar{\xi}_a}, \quad \bar{c}_a = -\frac{\delta W}{\delta \xi_a} \tag{II-12}$$

we make a Legendre transformation

$$\Gamma' = W - \int d^4x\, (J_\mu^a A_a^\mu + \bar{\xi}_a\, c_a + \bar{c}_a\, \xi_a$$

$$+ \bar{\eta}_\alpha\, \psi_\alpha + \bar{\psi}_\alpha\, \eta_\alpha) \tag{II-13}$$

which is a functional that generates proper amputated Green's functions.

The relations dual to Eq. (II-12) are

$$J_\mu^a = -\frac{\delta \Gamma'}{\delta A_a^\mu}$$

$$\xi_a = -\frac{\delta \Gamma'}{\delta \bar{c}_a}, \quad \bar{\xi}_a = \frac{\delta \Gamma'}{\delta c_a}$$

$$\eta_\alpha = -\frac{\delta \Gamma'}{\delta \bar{\psi}_\alpha}, \quad \bar{\eta}_\alpha = \frac{\delta \Gamma'}{\delta \psi_\alpha} \tag{II-14}$$

It is noted that the BRS transformations are nilpotent with respect to $\hat{A}_\mu$, $\hat{\psi}$, $\hat{\bar{\psi}}$, and $\hat{c}$, i.e.

$$\delta_\lambda^2 \hat{A}_\mu = \delta_\lambda^2 \hat{\psi} = \delta_\lambda^2 \hat{\bar{\psi}} = \delta_\lambda^2 \hat{c} = 0 \tag{II-15}$$

This has the consequence of leaving $\mathcal{L}_{c.s.}$ invariant. Therefore, a change of fields according to Eq. (II-6) will only change $\mathcal{L}_{s.}$ in Z. On the other hand, the value of Z should remain the same, because the Jacobian is unity. This leads to

$$\begin{aligned} 0 = &\int d\hat{A}\, d\hat{c}\, d\hat{\bar{c}}\, d\hat{\psi}\, d\hat{\bar{\psi}} \exp \{i \int d^4x \mathcal{L}\} \cdot \\ &\cdot \int d^4x \, \{-J_\mu^a [\hat{D}^\mu, \hat{c}]_a - \bar{n}\, g_o\, \hat{c}\, \hat{\psi} + g_o\, \hat{\bar{\psi}}\, \hat{c}\, n \\ &+ \frac{1}{\alpha_o} \partial_\mu \hat{A}_a^\mu \xi_a - \bar{\xi}_a \frac{1}{2} g_o \, [\hat{c}, \hat{c}]_a\} \end{aligned} \tag{II-16}$$

which is the same as

$$\begin{aligned} \int d^4x \, \{&\frac{\delta\Gamma'}{\delta A_\mu^a(x)} \frac{\delta\Gamma'}{\delta K_a^\mu(x)} + \frac{\delta\Gamma'}{\delta\psi(x)} \frac{\delta\Gamma'}{\delta\bar{m}(x)} + \frac{\delta\Gamma'}{\delta m(x)} \frac{\delta\Gamma'}{\delta\bar{\psi}(x)} \\ &+ \frac{\delta\Gamma'}{\delta c_a(x)} \frac{\delta\Gamma'}{\delta \ell_a(x)} + \frac{1}{\alpha_o} (\partial_\mu A^\mu)_a \frac{\delta\Gamma'}{\delta\bar{c}_a(x)} \} \end{aligned} \tag{II-17}$$

Using the same line of argument, we obtain an equation of motion for the ghost field by making a change of variable $\hat{\bar{c}} \to \hat{\bar{c}} + \delta\hat{\bar{c}}$

$$\begin{aligned} 0 = &\int d\hat{A}\, d\hat{c}\, d\hat{\bar{c}}\, d\hat{\psi}\, d\hat{\bar{\psi}} \; \exp \{i\, d^4x\, \mathcal{L}\} \\ &\cdot \{ \partial_\mu [\hat{D}^\mu, \hat{c}]_a - \xi_a\} \end{aligned} \tag{II-18}$$

or

$$\partial_\mu \frac{\delta\Gamma'}{\delta K^a_\mu} = \frac{\delta\Gamma'}{\delta\bar{c}_a} \tag{II-19}$$

This last equation, together with the following definition

$$\Gamma = \Gamma' + \frac{1}{2\alpha_o}(\partial_\mu A^\mu_a)^2 \tag{II-20}$$

gives us the fundamental W-T identities for QCD

$$0 = \int d^4x \left(\frac{d\Gamma}{\delta A^a_\mu}\frac{\delta\Gamma}{\delta K^\mu_a} + \frac{\delta\Gamma}{\delta\psi}\frac{\delta\Gamma}{\delta\bar{m}} + \frac{\delta\Gamma}{\delta m}\frac{\delta\Gamma}{\delta\bar{\psi}} + \frac{\delta\Gamma}{\delta c_a}\frac{\delta\Gamma}{\delta\ell_a}\right) \tag{II-21}$$

In a compact notation, these are written as

$$\Gamma * \Gamma = 0 \tag{II-22}$$

So far, we have been dealing with bare quantities. Their renormalization in the presence of external sources has been thoroughly discussed in the literature.[7,8] In brief, it is done multiplicatively according to

$$\hat{A}_\mu = \sqrt{Z}\,\hat{A}^R_\mu$$
$$\hat{c} = \sqrt{\tilde{Z}}\,\hat{c}^R$$
$$\hat{\psi} = \sqrt{Z_f}\,\hat{\psi}^R$$
$$g_o = (X/\sqrt{Z}\,\tilde{Z})\,g^R, \quad \alpha_o = Z\,\alpha^R \tag{II-23}$$

The sources for the primary fields are scaled in a manner to make $\mathcal{L}_s$ of Eq. (II-8) form invariant, i.e.

$$J_\mu = J^R_\mu/\sqrt{Z}$$
$$\xi_a = \xi^R_a/\sqrt{\tilde{Z}}, \quad \eta_a = \eta^R_a/\sqrt{Z_f} \tag{II-24}$$

The renormalized proper amputated Green's functions then have the desired normalization at the positions of the particles.

The composite operator vertices are renormalized so that N-T identities remain as they appear in Eq. (II-21). This is accomplished through

$$K_\mu^a = \sqrt{\tilde{Z}}\ (K_\mu^a)^R$$

$$L_a = \sqrt{Z}\ L_a^R$$

$$m = \sqrt{\frac{Z\ \tilde{Z}}{Z_f}}\ m^R \qquad \text{(II-25)}$$

We shall assume in the following that this renormalization program has been applied loopwise for the Green's functions. We shall drop the superscript R in the following sections; all operators there are understood to have been properly renormalized, unless specified otherwise.

III. PROOF OF FACTORIZATION VIA WARD-TAKAHASHI IDENTITIES

Let us suppose that the Lagrangian contains both light and heavy quarks. Our attention will be focussed on those Green's functions with only light quarks and gluons. For example, in $e^+e^- \to$ hadrons, when the energy of the virtual photon is less than, say, 2 GeV, the relevant heavy quark is c, while the light quarks are u,d,s. We also need the photon in this case; but since it will be introduced minimally, we may as well work with QCD alone at this formal level.

We shall use tilded quantities to denote those in the full theory and untilded to denote the corresponding ones in the light theory.

As said earlier, it will be presently assumed that the renormalized generating functionals have been constructed to all orders in loop expansion, i.e.

$$\tilde{\Gamma} = \sum_{n=0}^{\infty} \tilde{\Gamma}_{(n)} \qquad \text{(III-1)}$$

$$\Gamma = \sum_{n=0}^{\infty} \Gamma_{(n)} \qquad \text{(III-2)}$$

where the subscript (n) specifies the order of loops to which the quantities are calculated. These generating functionals have as their arguments classical fields and composite sources. However, in Eq. (III-1), we are to disregard all those terms which contain classical fields of the heavy quarks. This is in accordance with our intent to study Green's functions with light fields only. An immediate consequence is that ψ, $\bar{\psi}$, m, and $\bar{m}$ appearing in W-T identities of Eq. (II-21) pertain to light quarks only.

It is convenient to introduce source terms in the Lagrangian of the light theory

$$\int d^4x\, \mathcal{L}_{\text{light theory}} \to \int d^4x\, \mathcal{L}_{\text{light theory}} + \sum_i N_i\, O_i \qquad \text{(III-3)}$$

where N_i are global parameters and O_i are the operators appropriate for Eq. (I-1), which will be further elucidated later on. Then

$$\Gamma\,(O_i) = \frac{\delta}{\delta N_i}\,\Gamma\,\Big|_{N_i=0} \qquad \text{(III-4)}$$

Operator insertions as above require additional renormalization which is well understood[7,8]. Here we assume that this has been done and we will come back to this issue in the next section. In the following, to save writing, it will be assumed that N_i is set to zero after differentiation $\delta/\delta\, N_i$ has been applied.

Since Eq. (I-1) is independent of the number of external light lines, then it is a statement of the generating functional

$$\tilde{\Gamma} = \Gamma + \frac{1}{M^2} \sum_i C_i\, \frac{\delta}{\delta N_i}\,\Gamma \qquad \text{(III-5)}$$

Now, let us accept that Eq. (I-1) is true at the n-loop level, i.e.

$$\tilde{\Gamma}_{(n)} = \Gamma_{(n)} + \frac{1}{M^2} \sum_{j=1}^{n} \sum_{i} C^i_{(j)} \frac{\delta\Gamma_{(n-j)}}{\delta N_i} + O(\frac{1}{M^4}) \tag{III-6}$$

where $C^i_{(j)}$ is a set of coefficients calculated at the j-loop level. This is certainly true at the tree level with $C_{(0)} = 0$. We proceed to show that Eq. (I-2) holds also at n+1 loop, and therefore inductively it is true for all loops.

We first observe that the W-T identities are satisfied in loop expansion. At the n+1 loop level, we have

$$\sum_{k=0}^{n+1} \tilde{\Gamma}_{(k)} * \tilde{\Gamma}_{(n+1-k)} = 0, \tag{III-7}$$

and

$$\sum_{k=0}^{n+1} \Gamma_{(k)} * \Gamma_{(n+1-k)} = 0. \tag{III-8}$$

By extending the power counting argument in I, we are assured that the difference between $\tilde{\Gamma}_{(n+1)}$ and $\Gamma_{(n+1)}$ in low energy regime is of order $1/M^2$ or smaller, where M is the generic mass of the heavy quarks. We express this as

$$\tilde{\Gamma}_{(n+1)} = \Gamma_{(n+1)} + \frac{1}{M^2} \Delta\tilde{\Gamma}_{(n+1)}. \tag{III-9}$$

$\Delta\tilde{\Gamma}_{(n+1)}$ may depend on M in powers of $\ell n M$ and $1/M^2$.

Now, Eq. (III-7) is written as

$$0 = \tilde{\Gamma}_{(0)} * \tilde{\Gamma}_{(n+1)} + \tilde{\Gamma}_{(n+1)} * \tilde{\Gamma}_{(0)} + \sum_{k=1}^{n} \tilde{\Gamma}_{(k)} * \tilde{\Gamma}_{(n+1-k)} \tag{III-10}$$

which becomes, upon substituting Eq. (III-6) and Eq. (III-9)

$$0 = \Gamma_{(0)} * (\Gamma_{(n+1)} + \frac{1}{M^2} \Delta\tilde{\Gamma}_{(n+1)}) + (\Gamma_{(n+1)} + \frac{1}{M^2} \Delta\tilde{\Gamma}_{(n+1)}) * \Gamma_{(0)}$$

$$+ \sum_{k=1}^{n} (\Gamma_{(k)} + \frac{1}{M^2} \sum_{j=1}^{k} \sum_{i} C^i_{(j)} \frac{\delta\Gamma_{(k-j)}}{\delta N_i}) *$$

$$(\Gamma_{(n+1-k)} + \frac{1}{M^2} \sum_{j'=1}^{n+1-k} \sum_{i} C^i_{(j)} \frac{\delta\Gamma_{(n+1-k-j')}}{\delta N_i} \tag{III-11}$$

Using Eq. (III-8) and dropping $O(1/M^4)$ terms, this is simplified into

$$0 = \Gamma_{(0)} * \Delta\tilde{\Gamma}_{(n+1)} + \Delta\tilde{\Gamma}_{(n+1)} * \Gamma_{(0)}$$

$$+ \sum_{k=1}^{n} [\Gamma_{(k)} * \sum_{j=1}^{n+1-k} \sum_{i} C^i_{(j)} \frac{\delta\Gamma_{(n+1-k-j)}}{\delta N_i}$$

$$+ \sum_{j=1}^{k} \sum_{i} C^i_{(j)} \frac{\delta\Gamma_{(k-j)}}{\delta N_i} * \Gamma_{(n+1-k)}] \tag{III-12}$$

in which the sums can be rearranged

$$\sum_{k=1}^{n} \sum_{j=1}^{n+1-k} = \sum_{j=1}^{k} \sum_{k=1}^{n+1-j}$$

$$\sum_{k=1}^{n} \sum_{j=1}^{n} = \sum_{j=1}^{n} \sum_{k-j=0}^{k-j} \tag{III-13}$$

to yield

$$0 = \Gamma_{(0)} * \Delta\tilde{\Gamma}_{(n+1)} + \Delta\tilde{\Gamma}_{(n+1)} * \Gamma_{(0)}$$

$$+ \sum_{j=1}^{n} \sum_{i} C^{i}_{(j)} \left[\sum_{k=1}^{n+1-j} \Gamma_{(k)} * \frac{\delta\Gamma_{(n+1-k-j)}}{\delta N_i} \right.$$

$$\left. + \sum_{k=0}^{n-j} \frac{\delta\Gamma_{(k)}}{\delta N_i} * \Gamma_{(n+1-k-j)} \right] \qquad \text{(III-14)}$$

By differentiating Eq. (III-8) with respect to N_i and setting $N_i = 0$ and $n \to n-j$, we obtain

$$\sum_{k=0}^{n+1-j} \left(\frac{\delta\Gamma_{(k)}}{\delta N_i} * \Gamma_{(n+1-k-j)} + \Gamma_{(k)} * \frac{\delta\Gamma_{(n+1-k-j)}}{\delta N_i} \right) = 0 \qquad \text{(III-15)}$$

or

$$\sum_{k=1}^{n+1-j} \Gamma_{(k)} * \frac{\delta\Gamma_{(n+1-k-j)}}{\delta N_i} + \sum_{k=0}^{n-j} \frac{\delta\Gamma_{(k)}}{\delta N_i} * \Gamma_{(n+1-k-j)}$$

$$= - \left(\Gamma_{(0)} * \frac{\delta\Gamma_{(n+1-j)}}{\delta N_i} + \frac{\delta\Gamma_{(n+1-j)}}{\delta N_i} * \Gamma_{(0)} \right) \qquad \text{(III-16)}$$

This replacement transforms Eq. (III-14) into

$$0 = \Gamma_{(0)} * \left(\Delta\tilde{\Gamma}_{(n+1)} - \sum_{j=1}^{n} \sum_{i} C^{i}_{(j)} * \frac{\delta\Gamma_{(n+1-j)}}{\delta N_i} \right)$$

$$+ \left(\Delta\tilde{\Gamma}_{(n+1)} - \sum_{j=1}^{n} \sum_{i} C^{i}_{(j)} \frac{\delta\Gamma_{(n+1-j)}}{\delta N_i} \right) * \Gamma_{(0)} \qquad \text{(III-17)}$$

Let us go back to the defining equation of the short-hand notation (i.e. Eq. (II-22)) and write out various derivatives

$$\frac{\delta\Gamma_{(0)}}{\delta K^{\mu}_{a}} = -[D_{\mu},c]_{a},\ \frac{\delta\Gamma_{(0)}}{\delta\ell_{a}} = \frac{1}{2}\,g\,[c,c]_{a}$$

$$\frac{\delta\Gamma_{(0)}}{\delta\bar{m}} = g\,c\psi,\ \frac{\delta\Gamma_{(0)}}{\delta m} = g\,\bar{\psi}\,c \tag{III-18}$$

Then, Eq. (III-17) is written as

$$\int d^4x\,\mathcal{S}\,(\Delta\tilde{\Gamma}_{n+1} - \sum_{j=1}^{n}\sum_{i} C^{i}_{(j)}\,\frac{\delta\Gamma_{(n+1-j)}}{\delta N_i}) = 0 \tag{III-19}$$

where

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 \tag{III-20}$$

$$\mathcal{S}_0 = -[D_{\mu},c]_a\,\frac{\delta}{\delta A^{a}_{\mu}} + g\,c\,\psi_{\alpha}\,\frac{\delta}{\delta\psi_{\alpha}}$$

$$+ g\,\bar{\psi}_{\alpha}\,c\,\frac{\delta}{\delta\bar{\psi}_{\alpha}} + \frac{1}{2}\,g\,[c,c]_a\,\frac{\delta}{\delta c_a} \tag{III-21}$$

and

$$\mathcal{S}_1 = \frac{\delta\Gamma_{(0)}}{\delta A^{a}_{\mu}}\,\frac{\delta}{\delta K^{\mu}_{a}} + \frac{\delta\Gamma_{(0)}}{\delta\psi_{\alpha}}\,\frac{\delta}{\partial\bar{m}_{\alpha}}$$

$$+ \frac{\delta\Gamma_{(0)}}{\delta\bar{\psi}_{\alpha}}\,\frac{\delta}{\delta m_{\alpha}} + \frac{\delta\Gamma_{(0)}}{\delta c_a}\,\frac{\delta}{\delta\ell_a} \tag{III-22}$$

The nilpotency of $\mathcal{S}$ (i.e. $\mathcal{S}^2=0$) dictates that the local solutions[7,8] to Eq. (III-19) are either gauge invariant operators with densities O^{GI}_{i} or operators with densities $\mathcal{S}\,F_i$, where it is understood that the former cannot be written as the latter. F_i are some polynomial functionals of the classical fields and composite sources. Needless to say, these densities should possess the correct mass dimension, which is six in this context. This set of operators closes under renormalization.

As we promised, we have actually indentified what O_i should be in Eqs. (III-3,4). Since the operator basis in the solutions of Eq. (III-19) is independent of the loop number, they must then be trees. In other words, they can all be expressed as

$$O_i = \frac{\delta\Gamma_{(0)}}{\delta N_i} \tag{III-23}$$

It then follows that the coefficient functions multiplied to the solutions at the n+1 loop level must be evaluated at the same loop level, i.e.

$$\Delta\tilde{\Gamma}_{(n+1)} - \sum_{j=1}^{n} \sum_i C^i_{(j)} \frac{\delta\Gamma_{(n+1-j)}}{\delta N_i} = \sum_i C^i_{(n+1)} \frac{\delta\Gamma_{(0)}}{\delta N_i} \tag{III-24}$$

or

$$\Delta\tilde{\Gamma}_{(n+1)} = \sum_{j=1}^{n+1} \sum_i C^i_{(j)} \frac{\delta\Gamma_{(n+1-j)}}{\delta N_i} \quad . \tag{III-25}$$

This, when substituted into Eq. (III-8), completes the inductive proof

We would like to address the important issue of infrared and collinear singularities. As it is well-known, they appear in amplitudes as $\ln(p^2)$ for $p^2 \cong 0$, where p is some of the externla momenta. Hau we not been able to absorb them properly, then our analysis would have been in jeopardy. This is because operators with logarithmic vertex factors, if they exist at all, are simply not local.

In actuality, these infrared and collinear singularities are absorbed by the matrix elements of the operators, $\Gamma^n(O_i)$, or equivalently $\delta\Gamma^n/\delta N_i$. It works in such a way that both $\Delta\tilde{\Gamma}_{n+1}$ and $\sum_{i,j} C^i_{(j)}\ \delta\Gamma_{(n+1-j)}/\delta N_i$ in Eq. (III-19) may be plagued with these singularities. However, they cancel out completely in the difference, so that the solutions as given in Eq. (III-24) are local.

This pleasing situation arises, because infrared and collinear divergences occur when some of the internal lines become almost physical. These coincide with the low momentum and low invariant mass regions where effective vertices are introduced. In other words, $\sum_{i,j} C^i_{(j)}\ \delta\Gamma_{(n+1-j)}/\delta N_i$ contains all the integration regions which potentially may give soft divergences in $\Delta\tilde{\Gamma}_{(n+1)}$. The difference is then free of such malaise. Thereupon, the solutions of Eq. (III-19) are local, as given in Eq. (III-24).

IV. RENORMALIZATION OF O_i AND RENORMALIZATION GROUP FOR C_i

A. Renormalization of O_i

In the previous section, we have proved factorization, assuming the existence of some suitable procedure which accomplishes Eq. (III-6) in the renormalized form. In the proof, especially in going from Eq. (III-8) to Eq. (III-15), it was assumed that $\delta\Gamma_{(k)}/\delta N_i$ could be properly renormalized. That we have at hand such a program has been discussed by others.

We want to argue in the following that the counter terms introduced for operators O_i in fact cancel out internally; the relevant counter terms which need to be inserted into the Lagrangian are only those in Eqs. (II-23, 24,25). Because of this, the scheme for renormalizing the operators may be quite different from the scheme for making ordinary Green's functions finite.

Our assertion is true for a rather trivial reason. We illustrate it with an example. Consider the diagram of Fig. 1. Here, R indicates that ordinary renormalizations of Eq. (II-23) have been applied. Let us first project out the $1/M^2$ part from the vacuum polarization and extract out the associated operator vertices. Then, these operators are renormalized in whatever convenient way we wish. Let us donate this by R'. The original diagram will be redrawn in Fig. 2, where $C_{(1)}$ is the one loop coefficient function and O_1 is an induced operator. This is a mere rearrangement; clearly, the counter term due to R' simply cancels out and can be arbitrarily chosen.

Now, we look at what we denote by $\Delta_1\Sigma$. Any possible infrared or collinear divergence will cancel in this combination. (Actually, there is none in this example.) The $1/M^2$ part has a structure which is polynomial in momentum. We extract out the $1/M^2$ part and call the coefficient function $C_{(2)}$ and the induced operator O_2. This is depicted in Fig. 3. Note how this rearrangement scheme is precisely of the form dictated by the W-T identities of Eq. (III-24). Then, Fig. 1 is redrawn in Fig. 4. Due to the definition of $C_{(2)}/M^2$, $\Delta_2\Sigma$ goes as $1/M^4$ and can be dropped. Thus, we have completely factorized out the $1/M^2$ contributions in this example. Note that although $C_{(2)}$ depends on how we choose to renormalize the operator O_1, the sum $(C_{(1)}\Gamma_{(1)}(O_1) + C_{(2)}\Gamma_{(0)}(O_2))/M^2$ does not.

This argument can be easily extended to cover the general case. The important point is that since we are adding zero to $\tilde{\Gamma}$, we may split the zero into any two parts at our convenience.

B. Renormalization Group Equations For C_i

Factorization proved in Section III does not necessarily insure the calculability of the coefficient functions C_i, without which its usefulness is diminished. In the following, we shall show that C_i's obey a set of renormalization group equations and can be calculated.

Let us first recall that we are dealing with a theory with massless gluons and massless quarks. In order to avoid mass singularities and at the same time to assure the decoupling of heavy quarks, we are instructed to choose a subtraction point μ for Green's functions at some Euclidean point ($\mu^2 << M^2$).

This procedure necessarily introduces operators which have 'relevant' dimension four, although their naive mass dimension is six. For example, the operator $F^a_{\mu\nu}\,(\partial^2+\mu^2)F^{\mu\nu}_a$ has a piece $\mu^2\,\mathcal{O}_{42} = \mu^2(-\frac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu}_a)$, whose relevant dimension is four. We shall regard O_{42} as an independent operator. There are then two choices we can make with regard to its renormalization. If we take it as a dimension six operator, then oversubtraction is called for. On the other hand, we may count it as a dimension four operator, then the subtraction should be normal.

For the fermion operators, let us subtract such that chiral invariance is respected. Then, we shall induce an operator $\bar{\psi}((iD)^2-\mu^2)i\not{D}\psi$. Again, the part $\mu^2\mathcal{O}_{41}=-\mu^2\bar{\psi}\not{D}\psi$

may be regarded either as a dimension four or a dimension six operator in subtraction.

As we explained earlier, we are quite free to pick either of these alternatives. We find it convenient to regard operators like O_{41} and O_{42} as dimension four operators and subtract them normally. The reason is that otherwise when we come to decoupling the renormalization group equations, we have to express the oversubtracted operators in terms of the normally subtracted ones in order to form an independent basis. This effort is cumbersome and unnecessary.

Let $\Gamma^{B,F}$ be the light Green's functions with B external gluons and F external light quarks. $\tilde{\Gamma}^{B,F}$ will be similarly defined. Ghost external lines can be added, but let us not do so at this point. Then Eq. (I-1) is written as

$$\tilde{\Gamma}^{B,F} = \Gamma^{B,F} + \frac{1}{M^2} \sum_{N,a} C_{Na}\, \Gamma^{B,F}(O_{Na}) \qquad \text{(IV-1)}$$

where N denotes the dimension of the operators and a is an index to label different operators of the same dimension. We shall list and give a detailed discussion of these operators in a subsequent paper. For the present purpose, all we need to know is that $N = 4,6$. All operators will be normalized at the subtraction point μ to their tree vertex values.

It is important to note that the normalization conditions imposed on $\tilde{\Gamma}^{B,F}$ and $\Gamma^{B,F}$ will relate C_{4a}'s to C_{6b}'s in a simple way. For example, let us demand that $(\Gamma^{2,0},\tilde{\Gamma}^{2,0})$ and $(\Gamma^{0,2},\tilde{\Gamma}^{0,2})$ should satisfy, respectively, the same normalization conditions at the subtraction point. Then the coefficient function for $\frac{-i}{4} F_a^{\mu\nu}\partial^2 F_{\mu\nu}^a$ must be the same as that for $\mu^2 O_{42}$; likewise, the coefficient function for $i\bar{\psi}(iD)^2 i\not{D}\psi$ equals that of $\mu^2 O_{41}$. Because of this, we will only need to solve the equations for C_{6b}'s.

By standard argument, the scaling equations for the light theory are

$$(\mu \frac{\partial}{\partial\mu} + \beta_g \frac{\partial}{\partial g} + \beta_\alpha \frac{\partial}{\partial\alpha} - B\gamma_B - F\gamma_F)\Gamma^{B,F} = 0 \qquad \text{(IV-2)}$$

$$\{(\mu \frac{\partial}{\partial\mu} + \beta_g \frac{\partial}{\partial g} + \beta_\alpha \frac{\partial}{\partial\alpha} - B\gamma_B - F\gamma_F)\delta_{Ma,Nb} + \gamma_{Ma,Nb}\}\, \Gamma^{B,F}(O_{Nb}) = 0 \qquad \text{(IV-3)}$$

where we have taken the convention to sum over repeated indices and

$$\beta_g = \mu\frac{d}{d\mu}\, g, \qquad \beta_\alpha = \mu\frac{d}{d\mu}\, \alpha$$

$$2\gamma_F = \mu\frac{d}{d\mu}\, \ell n\, Z_F, \quad 2\gamma_B = \mu\,\frac{d}{d\mu}\, \ell n\, Z_B$$

$$\gamma_{Ma,Nb} = (Z\mu\, \frac{d}{d\mu}\, Z^{-1})_{Ma,Nb}\, (= 0\,,\ \text{for } M<N) \qquad \text{(IV-4)}$$

Z is the operator mixing matrix

$$\Gamma^{B,F}_{bare}\, (O^{bare}_{Ma}) = Z_B^{-B/2}\, Z_F^{-F/2}\, (Z^{-1})_{Ma,Nb}\, \Gamma^{B,F}(O_{Nb}) \qquad \text{(IV-5)}$$

For the full theory, we have

$$(\mu \frac{\partial}{\partial\mu} + \tilde{\beta}_g \frac{\partial}{\partial g} + \tilde{\beta}_M \frac{\partial}{\partial M} + \tilde{\beta}_\alpha \frac{\partial}{\partial\alpha} - B\tilde{\gamma}_B - F\tilde{\gamma}_F)\, \tilde{\Gamma}^{B,F} = 0 \qquad \text{(IV-6)}$$

where the anomalous dimensions are similarly defined. The only new symbol is

$$\beta_M = \mu \frac{d}{d\mu} M \qquad \text{(IV-7)}$$

which will give $1/M^4$ effects and hence will be ignored.

We substitute Eq. (IV-1) into Eq. (IV-6) and make use of Eqs. (IV-2,3). This gives

$$\frac{\mu^2}{M^2}(\Delta\beta_g \frac{\partial}{\partial g} + \Delta\beta_\alpha \frac{\partial}{\partial\alpha} - B\Delta\gamma_B - F\Delta\gamma_F)\Gamma^{B,F}$$
$$+ \frac{1}{M^2}\Gamma^{B,F}(O_{Ma})\ [(\mu \frac{\partial}{\partial\mu} + \beta_g \frac{\partial}{\partial g} + \beta_\alpha \frac{\partial}{\partial\alpha})\ \delta_{Ma,Nb}$$
$$-\gamma^t_{Ma,Nb}]\ C_{Nb} = 0 \qquad \text{(IV-8)}$$

where

$$\frac{\mu^2}{M^2}\Delta\beta_g = \tilde{\beta}_g - \beta_g, \text{ etc.} \qquad \text{(IV-9)}$$

and 't' stands for transposition of the matrix. Note that Eq. (IV-8) is an inhomogeneous equation for C_{Nb}'s due to the presence of the first part. Moreover, $\Delta\beta$'s and $\Delta\gamma$'s contain large $\ln$'s. Thus, as it stands, this equation is not particularly useful. The same apparent difficulty was encountered in I and was resolved with the use of certain counting identities. This method works here as well.

By a similar consideration as in I, we can easily prove the following counting identities[9]

$$\Gamma^{B,F}(O_{41}) = \frac{F}{2}\Gamma^{B,F}$$

$$\Gamma^{B,F}(O_{42}) = (\alpha\frac{\partial}{\partial\alpha} - \frac{g}{2}\frac{\partial}{\partial g} + \frac{B}{2})\Gamma^{B,F}$$

$$\Gamma^{B,F}(O_{43}) = g\frac{\partial}{\partial g}\Gamma^{B,F} + 2\Gamma^{B,F}(O_{42})$$

$$\Gamma^{B,F}(O_{44}) = 0 \qquad \text{(IV-10)}$$

where the newly introduced operators are

$$O_{43} = i\int d^4x\,(\frac{\partial S}{\partial A^a_\mu}A^a_\mu + \partial_\mu\bar{c}_a\,[D^\mu,c]_a)$$

$$O_{44} = i\int d^4x\,\partial_\mu\bar{c}_a\,[D^\mu,c]_a \qquad \text{(IV-11)}$$

S is the action corresponding to $\mathcal{L}_{eff}$ of Eq. (II-5) without the gauge fixing term. O_{41} through O_{44} are the only operators of dimension four which appear in the factorization formula. Now, the crucial point is that with the use of Eq. (IV-10), we can write the first part of Eq. (IV-8) as

$$\mu^2\ [\Delta\beta_g\frac{\partial}{\partial g} + \Delta\beta_\alpha\frac{\partial}{\partial\alpha} - B\Delta\gamma_B - F\Delta\gamma_F]$$
$$= C'_{4a}\,\Gamma^{B,F}(O_{4a}) \qquad \text{(IV-12)}$$

where C'_{4a} are independent of B and F. (They may be expressed in terms of $\Delta\beta$'s, $\Delta\gamma$'s, etc., but the details are not needed for our purpose.) Substituting Eq. (IV-12) into Eq. (IV-8), we obtain

$$\Gamma^{B,F}(O_{6a})\{(\mu \frac{\partial}{\partial\mu} + \beta_g \frac{\partial}{\partial g} + \beta_\alpha \frac{\partial}{\partial\alpha}) \delta_{6a,6b} - \gamma^t_{6a,6b}\} C_{6b}$$

$$+ \Gamma^{B,F}(O_{4a}) \{C'_{4a} + (\mu \frac{\partial}{\partial\mu} + \beta_g \frac{\partial}{\partial g} + \beta_\alpha \frac{\partial}{\partial\alpha}) C_{4a}$$

$$- \gamma^t_{4a,4b} C_{4b} - \gamma^t_{4a,6b} C_{6b}\} = 0 \qquad \text{(IV-13)}$$

With this form, we may invoke the independence of the operators and assert that the equations for C_{6b} decouple from the rest, i.e.

$$\{(\mu \frac{\partial}{\partial\mu} + \beta_g \frac{\partial}{\partial g} + \beta_\alpha \frac{\partial}{\partial\alpha}) \delta_{6a,6b} - \gamma^t_{6a,6b}\} C_{6b} = 0 \qquad \text{(IV-14)}$$

This is the desired set of equations.

We shall give in a separate paper an explicit calculation based on this work with application to $e^+e^- \to$ hadrons.

V. CONCLUSION

We have used W-T identities due to BRS transformations to prove the factorization formula Eq. (I-1) for QCD, which takes stock of heavy quark effects in low energy light quark and gluon physics. It is amusing to note the accomplished simplification, compared with I.

Our method gives a precise meaning to the notion of an effective Lagrangian; in particular, a well-prescribed renormalization procedure for the local operators is shown to be self generated by the theory. This is a major distinction over the naive approach, where one is nagged by issues of renormalizability of the effective interaction.

Note further that the coefficient functions or, equivalently, the effective couplings can be reliably calculated to any order of accuracy in an asymptotically free theory.[10] We need to account for only the nearby heavy quarks at any energy. The far away quarks will be suppressed by asymptotic freedom, in addition to the factor $1/M^2$.

In a companion article,[11] we shall give a detailed discussion of the structure of the operators and an explicit calculation of the anomalous dimensions to one loop order. The application to $e^+e^- \to$ hadrons will be used as an example of our approach. There, q^2 - the energy squared of the virtual photon - is quite large. We may compute operator matrix elements in powers of the running coupling constant $\bar{g}(q^2)$ via Eq. (IV-3).

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FIGURE CAPTIONS

Fig.1. A fourth order diagram whose heavy mass dependence is to be extracted. R denotes renormalization of the divergences of the Green's functions.

Fig.2. The heavy mass dependence is isolated from the vacuum polarization tensor in the form of $\frac{C_{(1)}}{M^2} O_1$. Note that the renormalization of O_1, denoted by R', can be differently chosen from R.

Fig.3. Further heavy mass dependence is isolated after $\frac{C_{(1)}}{M^2} O_1$ has been extracted. This is denoted by $\frac{C_{(2)}}{M^2} O_2$.

Fig.4. The fourth order graph of Fig. 1 is rearranged. $\Delta_2\Sigma$ can be discarded to the accuracy of $O(1/M^2)$.

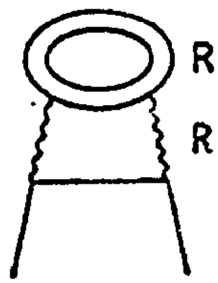

Fig. 1

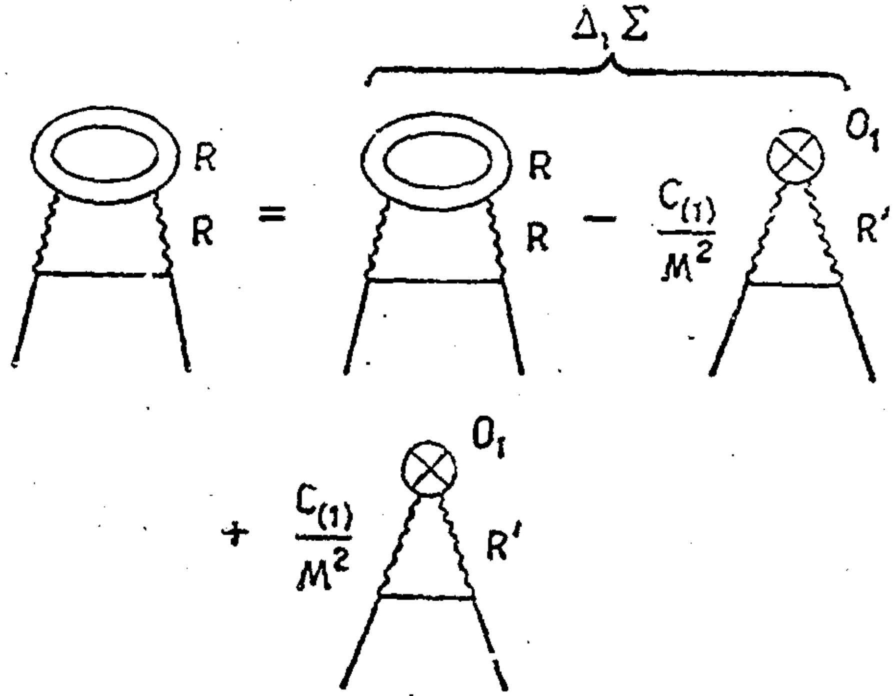

Fig. 2

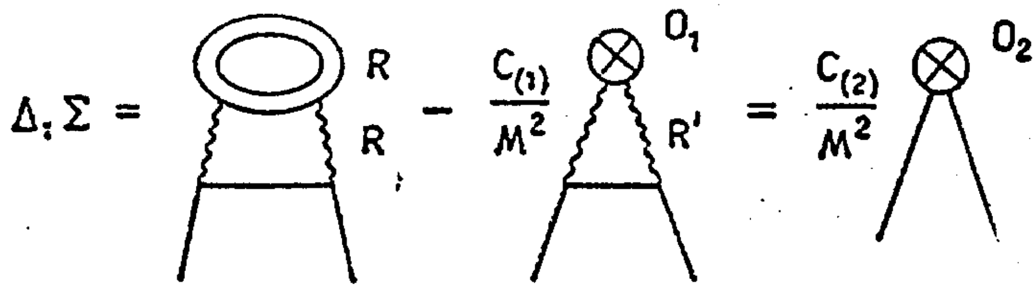

Fig. 3

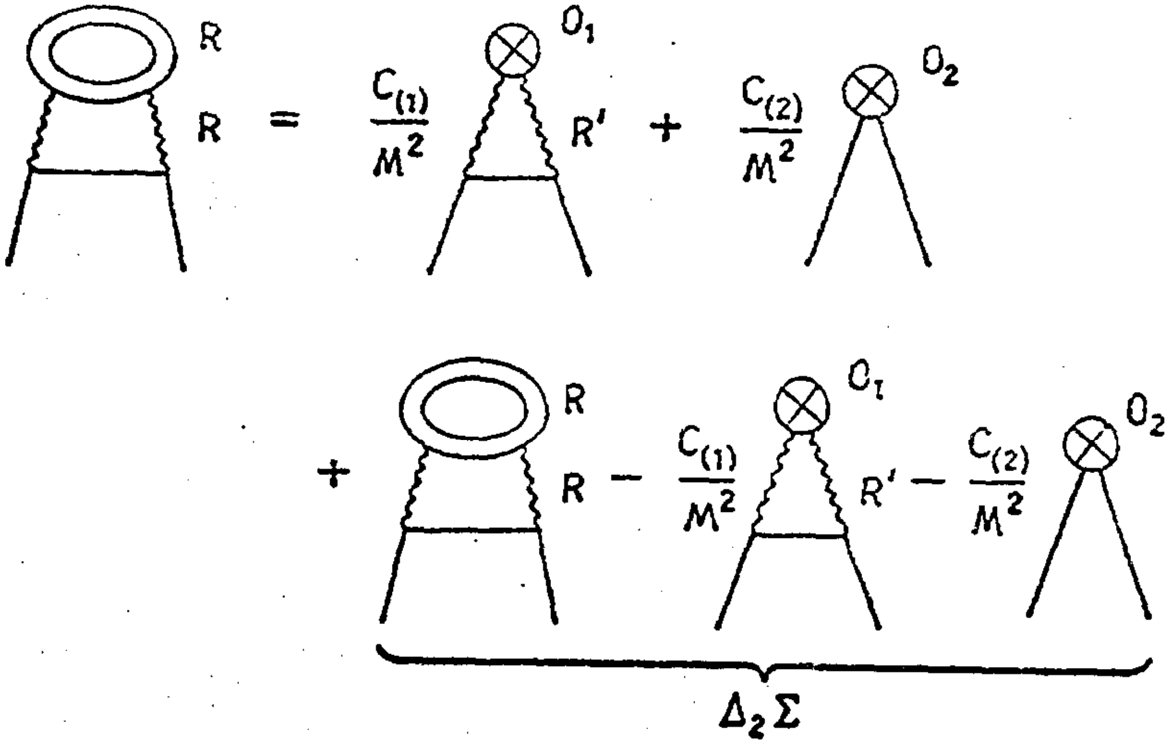
R
R
=
$\frac{C_{(1)}}{M^2}$
O_1
R'
+
$\frac{C_{(2)}}{M^2}$
O_2
+
R
R
−
$\frac{C_{(1)}}{M^2}$
O_1
R'
−
$\frac{C_{(2)}}{M^2}$
O_2
$\Delta_2 \Sigma$

Fig. 4